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On $\mathbb{F}_q$-linear sets of $PG(3,q^3)$ and semifields

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Abstract

Any finite semifield 2-dimensional over its left nucleus and $2n$-dimensional over its center defines a linear set of rank $2n$ of $PG(3,q^n)$ disjoint from a hyperbolic quadric and conversely [G. Lunardon, Translation ovoids, J. Geom. 76 (2003) 200–215]. Using this connection, semifields 2-dimensional over their left nucleus and 4-dimensional over their center were classified [I. Cardinali, O. Polverino, R. Trombetti, Semifield planes of order q^4 with kernel $\mathbb{F}_{q^2}$ and center $\mathbb{F}_q$, European J. Combin. 27 (2006) 940–961]. In this paper we give a characterization result in the case $n=3$, proving that there exist five or six non-isotopic families of such semifields, the families $\mathcal{F}_i$, $i=0,\dots,5$ ($\mathcal{F}_3$ might be empty), according to the different configurations of the associated linear sets of $PG(3,q^3)$. Also, we prove that to any semifield belonging to the family $\mathcal{F}_5$ is associated an $\mathbb{F}_q$-*pseudoregulus* of $PG(3,q^3)$ and we characterize the known examples of semifields of the family $\mathcal{F}_5$ in terms of the associated $\mathbb{F}_q$-pseudoregulus.

Keywords: Semifield; Linear set; Canonical subgeometry

1. Introduction

A *finite semifield* S is a finite algebraic structure containing at least two elements with two binary operations, addition and multiplication, which satisfy the following axioms.

A1. $(S,+)$ is a group, with identity element 0.
A2. If $ab=0$, then $a=0$ or $b=0$.

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A3. $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$ for any $a,b,c\in S$.
A4. There is an element 1 in S such that $1a=a1=a$ for all a in S.

Throughout this paper the term semifield will be always used to denote a finite semifield. The algebraic structure $(S,+,\cdot)$ is a pre-semifield if it satisfies all the axioms for a semifield, except possibly A4. A semifield can be constructed from a pre-semifield in several ways (see e.g. [8]).

Semifields coordinatize certain translation planes (called *semifield planes*) and two semifield planes are isomorphic if and only if the corresponding (pre)semifields are *isotopic* (see [8, §3]). A semifield is isotopic to a field if and only if the corresponding semifield plane is Desarguesian. Also, by using the André/Bruck and Bose construction [6], to any semifield are associated certain spreads (called *semifield spreads*). If S is a semifield, the following are subfields of S: the *left nucleus* $N_l=\{x\in S\colon (xa)b=x(ab),\ \forall a,b\in S\}$, the *middle nucleus* $N_m=\{x\in S\colon (ax)b=a(xb),\ \forall a,b\in S\}$, the *right nucleus* $N_r=\{x\in S\colon (ab)x=a(bx),\ \forall a,b\in S\}$, the *center* $K=\{x\in N_l\cap N_m\cap N_r\colon xy=yx,\ \forall y\in S\}$.

Let b be an element of the semifield S with center K; then the map $\phi_b : x\in S\to xb\in S$ is linear map when S is regarded as a left vector space over N_l. The set $\mathcal{C}_S=\{\phi_b\colon b\in S\}$ is closed under the sum of linear maps and $\lambda\varphi_b=\varphi_{\lambda b}$ for any $\lambda\in K$, i.e. $\mathcal{C}_S$ is a K-vector subspace of the vector space of the N_l-linear maps of S. Fix a basis of S over N_l; if $m=\dim_{N_l} S$, then the set $\mathcal{C}_S$ can be seen as a subset of the vector space $\mathbb{V}$ of the $(m\times m)$-matrices over N_l.

If $m=2$, then $\mathbb{V}$ is a 4-dimensional vector space over N_l and $\mathcal{C}_S$ is a K-vector subspace of $\mathbb{V}$ of dimension $2n$, where $n=\dim_K N_l$. If $K=\mathbb{F}_q$, then $N_l=\mathbb{F}_{q^n}$ and hence $\mathcal{C}_S$ defines in $\mathbb{P}=PG(\mathbb{V},\mathbb{F}_{q^n})=PG(3,q^n)$ an $\mathbb{F}_q$*-linear set* of rank $2n$, namely $L(S)=L_{\mathcal{C}_S}=\{\langle M\rangle_{\mathbb{F}_{q^n}}\colon M\in\mathcal{C}_S\setminus\{\mathbf{0}\}\}$. Since the linear maps defining $L(S)$ are non-singular, the linear set $L(S)$ is disjoint from the hyperbolic quadric $\mathcal{Q}=Q^+(3,q^n)$ of $\mathbb{P}$ with equation $\left|\begin{smallmatrix} X_0 & X_1\\ X_2 & X_3\end{smallmatrix}\right|=X_0X_3-X_1X_2=0$. Conversely, if L is an $\mathbb{F}_q$-linear set (with $\mathbb{F}_q$ as maximal subfield of linearity) of rank $2n$ of $\mathbb{P}=PG(\mathbb{V},\mathbb{F}_{q^n})$ disjoint from the hyperbolic quadric $\mathcal{Q}$, the algebraic structure $S=(\mathbb{F}_{q^n}\times\mathbb{F}_{q^n},+,\circ)$ with $(x_0,x_1)\circ(x_2,x_3)=(x_0,x_1)M_{x_2,x_3}$, where M_{x_2,x_3} is the unique matrix of L whose first row's entries are x_2 and x_3, is a (pre)semifield 2-dimensional over $N_l=\mathbb{F}_{q^n}\times\{0\}$ with center $\mathbb{F}_q\times\{0\}$ and $L(S)=L$ (see [15]). Also, S is isotopic to a field if and only if $L(S)$ is a line of $\mathbb{P}$. In [7], the authors proved the following result.

Theorem 1.1. [7, Theorem 2.1] *Let S_1 and S_2 be two semifields 2-dimensional over their left nuclei with center $\mathbb{F}_q$. Then they are isotopic if and only if the associated $\mathbb{F}_q$-linear sets of $\mathbb{P}=PG(3,q^n)$ are isomorphic with respect to the subgroup of $P\Gamma O^+(4,q^n)$ fixing the reguli of the hyperbolic quadric $\mathcal{Q}$ of $\mathbb{P}$.*

We say that a semifield 2-dimensional over the left nucleus with center $\mathbb{F}_q$ is of *scattered type* (or simply *scattered*) if the associated $\mathbb{F}_q$-linear set of $\mathbb{P}=PG(3,q^n)$ is scattered (see [5]).

In Section 3 we study scattered semifields proving that the translation dual of a scattered semifield is a scattered semifield. Also, we prove that the known examples of scattered semifields belong to the following families: Knuth semifields of type (17) and (19) and generalized twisted fields [8, p. 241, 243].

In [7], semifields 2-dimensional over the left nucleus and 4-dimensional over the center were classified. In this paper we focus on the case $n=3$, proving that there exist five or six non-isomorphic families of semifields 2-dimensional over the left nucleus and 6-dimensional over the center $\mathbb{F}_q$, which we call $\mathcal{F}_i$, $i=0,\dots,5$, according to the different configurations of the associated $\mathbb{F}_q$-linear sets of $\mathbb{P}=PG(3,q^3)$. Using [3,4,18] the families $\mathcal{F}_i$, $i=0,1,2$ are completely characterized: S belongs to the family $\mathcal{F}_0$ if and only if S is isotopic to a Generalized Dickson semifield with either $\alpha = \sigma = 1$ and $\beta = q$ or $\sigma = q$ and $\alpha = \beta = 1$ (see [8, p. 241]); S belongs to the family $\mathcal{F}_1$, if and only if $q = 3$ and S is associated with the Payne–Thas ovoid of $Q^+(4, 3^3)$; S belongs to the family $\mathcal{F}_2$, if and only if $q = 3$ and S is associated with the Ganley flock of $PG(3, 3^3)$.

In [10] with Norman L. Johnson we prove that *cyclic semifields* [2, p. 415] with our parameters belong to the family $\mathcal{F}_4$. So far, examples of semifields belonging to the family $\mathcal{F}_3$ are not known.

All the other known examples of semifields 2-dimensional over the left nucleus and 6-dimensional over the center are of scattered type and they belong to the family $\mathcal{F}_5$. We prove that to any such semifield S is associated an $\mathbb{F}_q$*-pseudoregulus* $\mathcal{L}(S)$ of $\mathbb{P} = PG(3, q^3)$ which is a set of $q^3 + 1$ pairwise disjoint lines with exactly two transversal lines which defines a *derivation set* as the pseudoregulus of $PG(3, q^2)$ (see Section 2.2). We characterize the known examples of semifields belonging to the family $\mathcal{F}_5$ in terms of the associated $\mathbb{F}_q$-pseudoreguli in the following way: S is isotopic to a semifield of Knuth (17) or (19) type if and only if the transversal lines of $\mathcal{L}(S)$ are contained in the quadric $\mathcal{Q}$; S is isotopic to a generalized twisted field (q odd) if and only if the transversal lines of $\mathcal{L}(S)$ are external lines to $\mathcal{Q}$ pairwise polar and $\mathcal{L}(S)$ is preserved by the polarity induced by $\mathcal{Q}$; finally, S is isotopic to a generalized twisted field (q even) if and only if the transversal lines of $\mathcal{L}(S)$ are external lines to $\mathcal{Q}$ pairwise polar and for any line r of $\mathcal{L}(S)$ the stabilizer $\bar{G}_{t,r}$ (where $\bar{G}$ is the linear group which leaves invariant the reguli of $\mathcal{Q}$ and t is a transversal of $\mathcal{L}(S)$) leaves $\mathcal{L}(S)$ invariant.

2. Preliminary results

A $(n-1)$-spread of a projective space $PG(m-1, q)$ is a family $\mathcal{S}$ of mutually disjoint subspaces of rank n such that each point of $PG(m-1, q)$ belongs to an element of $\mathcal{S}$. It has been proved by Segre [17] that $(n-1)$-spreads of $PG(m-1, q)$ exist if and only if $m = rn$. If $r > 2$, an $(n-1)$-spread $\mathcal{S}$ of $PG(rn-1, q)$ is said to be *normal* if it induces a spread in any subspace generated by two of its elements, i.e. if $A, B \in \mathcal{S}$, then any element of $\mathcal{S}$ either is disjoint from $T = \langle A, B\rangle$ or is contained in T. We say that an $(n-1)$-spread $\mathcal{S}$ of $PG(rn-1, q)$ is *Desarguesian* if either $r > 2$ and $\mathcal{S}$ is normal, or $r = 2$ and the associated translation plane obtained by André/Bruck and Bose representation is Desarguesian.

Up to isomorphisms, there is a unique Desarguesian $(n-1)$-spread of $PG(rn-1, q)$ which can be obtained as follows. Let $PG(r-1, q^n) = PG(V, \mathbb{F}_{q^n})$. Any point x of $PG(r-1, q^n)$ defines an $(n-1)$-dimensional subspace $P(x)$ of $PG(rn-1, q)$ and $\mathcal{D} = \{P(x)\colon x \in PG(r-1, q^n)\}$ is a Desarguesian spread of $PG(rn-1, q)$. If $r > 2$, the incidence structure Π_{r-1}, whose points are the elements of $\mathcal{D}$ and whose lines are the $(2n-1)$-dimensional subspaces of $PG(rn-1, q)$ joining two distinct elements of $\mathcal{D}$, is isomorphic to $PG(r-1, q^n)$. The pair $(PG(rn-1, q), \mathcal{D})$ is called the *linear representation* of $PG(r-1, q^n)$ over $\mathbb{F}_q$. A Desarguesian $(n-1)$-spread of $PG(rn-1, q)$ can also be constructed as follows. Let $\Sigma = PG(rn-1, q)$. Embed Σ in $\Sigma^* = PG(rn-1, q^n)$ in such a way that Σ is the set of fixed points of a semilinear collineation σ of Σ^* of order n. Let $\Theta = PG(r-1, q^n)$ be a subspace of Σ^* such that Θ, $\Theta^\sigma, \ldots, \Theta^{\sigma^{n-1}}$ generate the whole space Σ^*. If x is a point of Θ, $\Pi(x) = \langle x, x^\sigma, \ldots, x^{\sigma^{n-1}}\rangle$ is an $(n-1)$-dimensional subspace of Σ^* defining an $(n-1)$-dimensional subspace $\pi(x) = \Pi(x) \cap \Sigma$ of Σ; as x varies over the subspace Θ we get a set of $q^{n(r-1)} + q^{n(r-2)} + \cdots + q^n + 1$ mutually disjoint $(n-1)$-subspaces of Σ. Such a set is denoted by $\mathcal{S}(\Theta, \ldots, \Theta^{\sigma^{n-1}})$ and it turns out to be a Desarguesian $(n-1)$-spread of Σ (see [17]).

Remark 2.1. Note that, since a Desarguesian $(n-1)$-spread of $PG(rn-1,q)$ is unique up to isomorphisms, if Θ' is an $(r-1)$-dimensional subspace of $PG(rn-1,q^n)$ such that $\Theta',\dots,\Theta'^{\sigma^{n-1}}$ span the whole space, then $\mathcal{S}(\Theta,\dots,\Theta^{\sigma^{n-1}})$ is isomorphic to $\mathcal{S}(\Theta',\dots,\Theta'^{\sigma^{n-1}})$ and hence there exists a collineation of Σ^* fixing Σ and mapping Θ to Θ'.

2.1. Linear sets

Let $\Omega = PG(r-1,q^n) = PG(V,\mathbb{F}_{q^n})$, $q=p^h$, p prime, and let L be a set of points of Ω. The set L is said to be an $\mathbb{F}_q$-*linear* set of Ω if it is defined by the non-zero vectors of an $\mathbb{F}_q$-vector subspace U of V, i.e., $L = L_U = \{\langle \mathbf{u}\rangle_{\mathbb{F}_{q^n}}\colon \mathbf{u}\in U\setminus\{\mathbf{0}\}\}$. If $\dim_{\mathbb{F}_q} U = t$, we say that L has *rank* t. Let $PG(rn-1,q) = PG(V,\mathbb{F}_q)$ and let $(PG(rn-1,q),\mathcal{D})$ be the linear representation of Ω over $\mathbb{F}_q$. A t-dimensional $\mathbb{F}_q$-vector subspace U of V defines in $PG(rn-1,q)$ a $(t-1)$-dimensional projective subspace $P(U)$ and the linear set L_U of Ω can be seen as the set of points x of Ω such that $P(x)\cap P(U)\neq\emptyset$, i.e. $L_U=\{x\in\Omega\colon\ P(x)\cap P(U)\neq\emptyset\}$. If $\Lambda = PG(W,\mathbb{F}_{q^n})$ is a subspace of Ω and L_U is an $\mathbb{F}_q$-linear set of Ω, then $\Lambda\cap L_U$ is an $\mathbb{F}_q$-linear set of Λ defined by the $\mathbb{F}_q$-vector subspace $U\cap W$. If L_U is an $\mathbb{F}_q$-linear set of Ω of rank t, we say that a point $x=\langle\mathbf{u}\rangle_{\mathbb{F}_{q^n}}$, $\mathbf{u}\in U$, of L_U has *weight* i in L_U if $P(U)$ intersects the spread element $P(x)$ in an $(i-1)$-dimensional projective subspace, i.e. $\dim_{\mathbb{F}_q}(\langle\mathbf{u}\rangle_{\mathbb{F}_{q^n}}\cap U)=i$, and we write $\omega(x)=i$. If x_i is the number of points of L of weight i, we have

$$|L| = x_1+x_2+\dots+x_n, \tag{1}$$

$$x_1+(q+1)x_2+\dots+\left(q^{n-1}+\dots+q+1\right)x_n = q^{t-1}+\dots+q+1. \tag{2}$$

Hence $|L|\leqslant q^{t-1}+\dots+q+1$. An $\mathbb{F}_q$-linear set L_U of Ω is *scattered* if all of its points have weight 1, or equivalently, if the projective subspace $P(U)$ of $PG(rn-1,q)$ intersects each element of $\mathcal{D}$ in at most one point (in [5] such a linear set is called *scattered* with respect to the Desarguesian spread $\mathcal{D}$). Naturally, if L_U is a scattered $\mathbb{F}_q$-linear set of $PG(r-1,q^n)$ of rank t then $|L_U| = |P(U)| = q^{t-1}+\dots+q+1$. In [5], the authors prove the following result.

Theorem 2.2. [5, Theorem 4.2] *If rn is even and L_U is a scattered $\mathbb{F}_q$-linear set of $PG(r-1,q^n)$ of rank $\frac{rn}{2}$, then each hyperplane π_{r-2} of $PG(r-1,q^n)$ intersects L_U in an $\mathbb{F}_q$-linear set of rank either $\frac{rn}{2}-n$ or $\frac{rn}{2}-n+1$, i.e. the size of $L_U\cap\pi_{r-2}$ is either $q^{\frac{rn}{2}-n-1}+\dots+q+1$ or $q^{\frac{rn}{2}-n}+\dots+q+1$.*

We say that a correlation d of $PG(rn-1,q)$ *preserves* the linear representation $(PG(rn-1,q),\mathcal{D})$ of $PG(r-1,q^n)$ if d maps any element of $\mathcal{D}$ to an $((r-1)n-1)$-dimensional projective subspace of $PG(rn-1,q)$ which is a union of elements of $\mathcal{D}$. A correlation d preserving $(PG(rn-1),\mathcal{D})$ induces a correlation d' of $PG(r-1,q^n)$. If d is a correlation preserving the linear representation of $PG(r-1,q^n)$ and L_U is an $\mathbb{F}_q$-linear set of $PG(r-1,q^n)$ of rank t, the *dual* of L_U with respect to d is the $\mathbb{F}_q$-linear set of $PG(r-1,q^n)$ defined by the $(rn-1-t)$-dimensional subspace $P(U)^d$ of $PG(rn-1,q)$. We denote by L_U^d the dual of L_U with respect to d. It is easy to prove that if Λ_i is an i-dimensional projective subspace of $PG(r-1,q^n)$, then

$$\dim_{\mathbb{F}_q}\left(\Lambda_i^{d'}\cap L_U^d\right) = rn-t-(i+1)n+\dim_{\mathbb{F}_q}(\Lambda_i\cap L_U). \tag{3}$$

The following property holds.

Proposition 2.3. *If rn is even and L_U is a scattered $\mathbb{F}_q$-linear set of $PG(r-1,q^n)$ of rank $\frac{rn}{2}$, then L_U^d is a scattered $\mathbb{F}_q$-linear set of rank $\frac{rn}{2}$.*

Proof. Let P be a point of $PG(r-1,q^n)$ and let $P=\pi^{d'}_{r-2}$ then, by (3), $\dim_{\mathbb{F}_q}(P\cap L_U^d)=\dim_{\mathbb{F}_q}(\pi^{d'}_{r-2}\cap L_U^d)=n-\frac{rn}{2}+\dim_{\mathbb{F}_q}(\pi_{r-2}\cap L_U)$. The proof now follows from Theorem 2.2. □

If $\dim_{\mathbb{F}_q} U=\dim_{\mathbb{F}_{q^n}} V=r$ and $\langle U\rangle_{\mathbb{F}_{q^n}}=V$, then the $\mathbb{F}_q$-linear set $L_U\cong PG(U,\mathbb{F}_q)$ is a *canonical subgeometry* of $PG(V,\mathbb{F}_{q^n})$.

In [14], the authors, generalizing results of [12,13], give the following characterization of $\mathbb{F}_q$-linear sets.

Let $\Sigma=PG(m,q)$ be a canonical subgeometry of $\Sigma^*=PG(m,q^n)$. Suppose there is an $(m-r)$-dimensional subspace Λ^* of Σ^* disjoint from Σ. Let $\Lambda=PG(r-1,q^n)$ be an $(r-1)$-dimensional subspace of Σ^* disjoint from Λ^*, and let

$$\Gamma=\{x \text{ is a point of } \Lambda \mid \exists y\in\Sigma\colon\ x=\langle\Lambda^*,y\rangle\cap\Lambda\}$$

be the *projection* of Σ from Λ^* onto Λ. We call Λ^* and Λ respectively the *center* and the *axis* of the projection.

Let $p_{\Lambda^*,\Lambda,\Sigma}$ be the map from Σ to Γ defined by $x\mapsto\langle\Lambda^*,x\rangle\cap\Lambda$ for each point x of Σ. By definition $p_{\Lambda^*,\Lambda,\Sigma}$ is surjective and $\Gamma=p_{\Lambda^*,\Lambda,\Sigma}(\Sigma)$.

Remark 2.4. Note that the projection Γ of a canonical subgeometry Σ does not depend on the choice of the axis Λ.

Theorem 2.5. [14, Theorem 1] *If Γ is a projection of $PG(m,q)$ onto $\Lambda=PG(r-1,q^n)$, then Γ is an $\mathbb{F}_q$-linear set of Λ of rank $m+1$ and $\langle\Gamma\rangle=\Lambda$.*

Conversely, we have

Theorem 2.6. [14, Theorem 2] *If L is an $\mathbb{F}_q$-linear set of Λ of rank $m+1$ and $\langle L\rangle=\Lambda=PG(r-1,q^n)$, then either L is a canonical subgeometry of Λ or for any $(m-r)$-dimensional subspace Λ^* of $\Sigma^*=PG(m,q^n)$ disjoint from Λ there exists a canonical subgeometry Σ of Σ^* disjoint from Λ^* such that $L=p_{\Lambda^*,\Lambda,\Sigma}(\Sigma)$.*

2.2. $\mathbb{F}_q$-pseudoreguli of $PG(3,q^3)$

Let L be a scattered $\mathbb{F}_q$-linear set of $\mathbb{P}=PG(3,q^3)$ of rank 6 and embed $\mathbb{P}$ in $\Sigma^*=PG(5,q^3)$. From Theorem 2.6, for any line ℓ of Σ^* disjoint from $\mathbb{P}$ there exists a canonical subgeometry $\Sigma=PG(5,q)$ of Σ^* disjoint from ℓ such that $L=p_{\ell,\mathbb{P},\Sigma}(\Sigma)$. Let ω denote a semilinear collineation of Σ^* of order 3 fixing pointwise the subgeometry Σ. Since L is a scattered $\mathbb{F}_q$-linear set, then it is easy to see that for each point x of ℓ the subspace $\Pi(x)=\langle x,x^\omega,x^{\omega^2}\rangle$ is a plane of Σ^*, i.e. ℓ, ℓ^ω and ℓ^{ω^2} span the whole space Σ^*. Hence the set $\mathcal{S}(\ell,\ell^\omega,\ell^{\omega^2})=\{\pi(x)=\Pi(x)\cap\Sigma\colon\ x\in\ell\}$ is a Desarguesian 2-spread of Σ. This implies that

Proposition 2.7. *Scattered $\mathbb{F}_q$-linear sets of $\mathbb{P}=PG(3,q^3)$ of rank 6 are projectively equivalent.*

Proof. Let L and L' be scattered $\mathbb{F}_q$-linear sets of $\mathbb{P}=PG(3,q^3)$ of rank 6. By Theorem 2.6 they are projection of canonical subgeometries Σ and Σ' of $\Sigma^*=PG(5,q^3)$ from the lines l and l', respectively. Since canonical subgeometries over the same field and of the same dimension are isomorphic and since the projection does not depend on the axis (Remark 2.4), we can suppose that $\Sigma = \Sigma'$. Let ω be a semilinear collineation of Σ^* of order 3 fixing Σ. Since $\mathcal{S}(l, l^\omega, l^{\omega^2})$ and $\mathcal{S}(l', l'^\omega, l'^{\omega^2})$ are Desarguesian 2-spreads of Σ by Remark 2.1 there exists a collineation ϕ of Σ^* fixing Σ sending l to l'. Then ϕ maps $L = p_{l,\mathbb{P},\Sigma}(\Sigma)$ to $L'' = p_{l',\phi(\mathbb{P}),\Sigma}(\Sigma)$ and since the projection does not depend on the axis then $L'' \cong L' = p_{l',\mathbb{P},\Sigma}(\Sigma)$, so L and L' are projectively equivalent. □

By Remark 2.4, we can suppose that $\mathbb{P} = \langle \ell^\omega, \ell^{\omega^2} \rangle$. Let x be a point of ℓ and denote by ℓ_x the line obtained intersecting the plane $\Pi(x)$ with $\mathbb{P}$, i.e. $\ell_x = \langle x^\omega, x^{\omega^2} \rangle$. The line ℓ_x contains the $q^2 + q + 1$ points of L which are the projection of $\pi(x)$ from ℓ to $\mathbb{P}$. As x varies over the line ℓ we get a set $\mathcal{L} = \{\ell_x\colon x \in L\}$ of $q^3 + 1$ mutually disjoint lines of $\mathbb{P}$ having the lines ℓ^ω and ℓ^{ω^2} as transversals and partitioning the linear set L. Also, L is disjoint from the lines ℓ^ω and ℓ^{ω^2} and the lines ℓ_x's are all the lines of $\mathbb{P}$ which meet L in $q^2 + q + 1$ points.

By Proposition 2.7, we can suppose that

$$L = \{(x, y, x^q, y^q)\colon x, y \in \mathbb{F}_{q^3}\}.$$

In this case the lines of $\mathcal{L}$ are $r_a\colon x_1 - a x_0 = x_3 - a^q x_2 = 0$, where $a \in \mathbb{F}_{q^3} \cup \{\infty\}$ and the two transversals are $t\colon x_2 = x_3 = 0$ and $t'\colon x_0 = x_1 = 0$. The point-set of $\mathcal{L} \setminus \{t, t'\}$ can be partitioned into $q - 1$ mutually disjoint scattered $\mathbb{F}_q$-linear sets of rank 6 as follows. Let $c \in \mathbb{F}_{q^3}^*$ and let

$$L_c = \{(x, y, c x^q, c y^q)\colon x, y \in \mathbb{F}_{q^3}\}.$$

Note that $L_c \cap L_d = \emptyset$ if $N_{\mathbb{F}_q}(c) \neq N_{\mathbb{F}_q}(d)$[1] and $L_c = L_d$ otherwise. Hence $\bigcup_{c \in \mathbb{F}_{q^3}^*} L_c = \mathcal{L} \setminus \{t, t'\}$. From Ostrom theory of derivation, it follows that the set of lines $\mathcal{L}$ defines a *derivation set* of a Desarguesian spread (see e.g. [16]). In analogy to the pseudoregulus of $PG(3, q^2)$ introduced by Freeman in [9], we will refer to the line-set $\mathcal{L}$ as an $\mathbb{F}_q$*-pseudoregulus* of $PG(3, q^3)$. Hence we have proved the following

Proposition 2.8. *To any scattered $\mathbb{F}_q$-linear set L of rank 6 of $\mathbb{P} = PG(3, q^3)$ is associated an $\mathbb{F}_q$-pseudoregulus $\mathcal{L}$ of $\mathbb{P}$ consisting of all $(q^2 + q + 1)$-secant lines of L.*

3. Semifields 2-dimensional over their left nucleus

In the geometric setting of the Introduction, let S be a semifield 2-dimensional over the left nucleus with center $\mathbb{F}_q$ and let $L(S)$ be the $\mathbb{F}_q$-linear set defined by $\mathcal{C}_S$ in $\mathbb{P} = PG(3, q^n)$ ($n = \dim_{\mathbb{F}_q} N_l$).

Let $(PG(4n - 1, q), \mathcal{D})$ be the linear representation of $\mathbb{P} = PG(3, q^n)$ over $\mathbb{F}_q$. If $b(X_0, X_1, X_2, X_3) = X_0 X_3 - X_1 X_2$ is the quadratic form which defines the quadric $\mathcal{Q} = Q^+(3, q^n)$, then $\mathrm{Tr}_{\mathbb{F}_q}(b(X_0, X_1, X_2, X_3)) = 0$[2] defines a non-singular quadric $\hat{\mathcal{Q}} = Q^+(4n - 1, q)$ of $PG(4n - 1, q)$. If l is a line contained in $\mathcal{Q}$, then the $(2n - 1)$-dimensional projective subspace of $PG(4n - 1, q)$ defined by l is a maximal singular subspace contained in $\hat{\mathcal{Q}}$. Let ρ denote the polarity of $PG(4n - 1, q)$ induced by $\hat{\mathcal{Q}}$. Such a polarity preserves the linear representation of

[1] $N_{\mathbb{F}_q}$ denotes the norm function of $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$.

[2] $\mathrm{Tr}_{\mathbb{F}_q}$ denotes the trace function of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

$\mathbb{P}$ and it induces on $\mathbb{P}$ the polarity $\perp$ of $\mathcal{Q}$. Let $L(S)^\rho$ be the $\mathbb{F}_q$-linear set representing the dual of $L(S)$ with respect to ρ (see Subsection 2.1). Note that $L(S)^\rho$ has rank $2n$ and, by (3), for any $x \in \mathcal{Q}$ we get

$$\dim_{\mathbb{F}_q}\bigl(x \cap L(S)^\rho\bigr) = \dim_{\mathbb{F}_q}\bigl(\bigl(x^\perp\bigr)^\perp \cap L(S)^\rho\bigr) = \dim_{\mathbb{F}_q}\bigl(x^\perp \cap L(S)\bigr) - n.$$

Since $L(S) \cap \mathcal{Q} = \emptyset$, then $\dim_{\mathbb{F}_q}(L(S) \cap x^\perp) = n$ for any tangent plane $x^\perp$ to $\mathcal{Q}$. Hence, the $\mathbb{F}_q$-linear set $L(S)^\rho$ is disjoint from the hyperbolic quadric $\mathcal{Q}$ and it defines a semifield 2-dimensional over the left nucleus with center $\mathbb{F}_q$, $S^\perp$, called the *translation dual* of S [15].

It can be easily seen that a semifield 2-dimensional over the left nucleus is symplectic if and only if the associated linear set $L(S)$ is contained in a plane of $\mathbb{P} = PG(3, q^n)$. On the other hand, a semifield S (2-dimensional over the left nucleus) is associated with a flock if and only if there exists a point of $L(S)$ of weight n, or equivalently, $L(S)$ is a union of lines through a point. So, a semifield S is symplectic if and only if its translation dual $S^\perp$ is associated with a flock (for more details, see [15]).

3.1. Semifields of scattered type

Let S be a semifield 2-dimensional over $N_l \cong \mathbb{F}_{q^n}$ with center $\mathbb{F}_q$ and let $S^\perp$ be the translation dual of S. Also, let $L(S)$ and $L(S^\perp)$ be the $\mathbb{F}_q$-linear sets associated with S and $S^\perp$, respectively. From Proposition 2.3 it follows

Proposition 3.1. *A semifield S is scattered if and only if $S^\perp$ is scattered.*

In the following we list the known examples of (pre)semifields of scattered type:

(I) *Knuth semifields* $K_1 = (\mathcal{F}, +, \circ)$ (see [8, p. 241 (multiplication (17))]), with $\mathcal{F} = \mathbb{F}_{q'} \times \mathbb{F}_{q'}$, $N_\ell = \mathbb{F}_{q'} \times \{0\}$ and

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1)\begin{pmatrix} x_2 & x_3 \\ f x_3^\sigma & x_2^\sigma + x_3^\sigma g \end{pmatrix},$$

where $\sigma \in \mathrm{Aut}(\mathbb{F}_{q'})$, with $\sigma \neq 1$, and f and g are non-zero elements in $\mathbb{F}_{q'}$ such that the polynomial $x^{q+1} + gx - f$ is irreducible over $\mathbb{F}_{q'}$. The linear set associated with the semifield K_1 is

$$L_{K_1} = \bigl\{\bigl(x, y, f y^\sigma, x^\sigma + g y^\sigma\bigr)\colon x, y \in \mathbb{F}_{q'}\bigr\}. \tag{4}$$

Note that the center of K_1 is $\mathbb{F}_q = \mathrm{Fix}(\sigma)$ $(q' = q^n)$, hence L_{K_1} is an $\mathbb{F}_q$-linear set of $\mathbb{P}$ of rank $2n$.

(II) *Knuth semifields* $K_2 = (\mathcal{F}, +, \circ)$ (see [8, p. 241 (multiplication (19))]), with $\mathcal{F} = \mathbb{F}_{q'} \times \mathbb{F}_{q'}$, $N_\ell = \mathbb{F}_{q'} \times \{0\}$ and

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1)\begin{pmatrix} x_2 & x_3 \\ f x_3^{\sigma^{-1}} & x_2^\sigma + x_3 g \end{pmatrix},$$

where $\sigma \in \mathrm{Aut}(\mathbb{F}_{q'})$, with $\sigma \neq 1$, f and g are non-zero elements in $\mathbb{F}_{q'}$ such that the polynomial $x^{q+1} + gx - f$ is irreducible over $\mathbb{F}_{q'}$. The linear set associated with the semifield K_2 is

$$L_{K_2} = \bigl\{\bigl(x, y, f y^{\sigma^{-1}}, x^\sigma + g y\bigr)\colon x, y \in \mathbb{F}_{q^n}\bigr\}. \tag{5}$$

Remark 3.2. Note that the Knuth semifields K_2 are the transpose semifields of the Knuth semifields K_1 (see [11, §5]).

(III) *Generalized twisted fields* $\mathcal{F}_c = (\mathcal{F}, +, \circ)$ (see [8, p. 243]), $\mathcal{F} = \mathbb{F}_{q'^2}$, with dimension 2 over their left nucleus $\mathbb{F}_{q'}$. The (pre)semifield multiplication is

$$x \circ y = xy - cx^{q'} y^{\beta}, \tag{6}$$

with $\beta \in \mathrm{Aut}(\mathbb{F}_{q'^2})$ and $c \in \mathbb{F}_{q'^2}$, such that $c \neq x^{q'-1} y^{\beta-1}$ for any $x, y \in \mathbb{F}_{q'^2}$.

Let q' be odd and take $\lambda \in \mathbb{F}_{q'^2}$ such that $\lambda^2 = \sigma$ where σ is a non-square element of $\mathbb{F}_{q'}$. Hence $c = c_1 + \lambda c_2$, with c_1 and $c_2 \in \mathbb{F}_{q'}$ and the (pre)semifield multiplication can be written in coordinate terms with respect to the $\mathbb{F}_{q'}$-basis $\{1, \lambda\}$ of $\mathbb{F}_{q'^2}$ as

$$(x_0, x_1) \circ (x_2, x_3) = (x_1, x_2) \begin{pmatrix} x_2 - c_1 x_2^{\beta} - \sigma c_2 a x_3^{\beta} & x_3 - c_1 a x_3^{\beta} - c_2 x_2^{\beta} \\ \sigma x_3 + a c_1 \sigma x_3^{\beta} + \sigma c_2 x_2^{\beta} & x_2 + c_1 x_2^{\beta} + \sigma c_2 a x_3^{\beta} \end{pmatrix},$$

where a is the element of $\mathbb{F}_{q'}$ such that $\lambda^{\beta} = a\lambda$ (in particular we have $a^2 = \sigma^{\beta-1}$). The linear set associated with $\mathcal{F}_c$ is

$$L_c = \big\{ \big(x - c_1 x^{\beta} - \sigma c_2 a y^{\beta}, y - c_1 a y^{\beta} - c_2 x^{\beta}, \sigma y + a c_1 \sigma y^{\beta} + \sigma c_2 x^{\beta}, \\ x + c_1 x^{\beta} + \sigma c_2 a y^{\beta}\big) \colon x, y \in \mathbb{F}_{q'} \big\}. \tag{7}$$

The maximal subfield of linearity of L_c is $\mathbb{F}_q = \mathrm{Fix}(\beta) \cap \mathbb{F}_{q'}$ $(q' = q^n)$. Hence, L_c is an $\mathbb{F}_q$-linear set of rank $2n$ of $\mathbb{P}$.

Let q' be even and take $\lambda \in \mathbb{F}_{q'^2} \setminus \mathbb{F}_{q'}$ such that $\lambda^2 + \lambda + \sigma = 0$, with $\sigma \in \mathbb{F}_{q'}$. Setting $\mathbb{F}_q = \mathrm{Fix}(\beta) \cap \mathbb{F}_{q'}$ $(q' = q^n)$, if $\bar{c} \in \mathbb{F}_{q'^2}$ such that $\bar{c} \neq x^{q'-1} y^{\beta-1}$, i.e., $N_{\mathbb{F}_q}(\bar{c}) \neq 1$, then it is easy to prove that there exists an element $c \in \mathbb{F}_{q'}$ such that $N_{\mathbb{F}_q}(\bar{c}) = N_{\mathbb{F}_q}(c)$. Hence, the (pre)semifields $\mathcal{F}_c$ and $\mathcal{F}_{\bar{c}}$ are isotopic [1, Theorem 1] and the (pre)semifield multiplication (6), with $c \in \mathbb{F}_{q'}$ can be written in coordinate terms as

$$(x_0, x_1) \circ (x_2, x_3) = (x_0, x_1) \begin{pmatrix} x_2 + c x_2^{\beta} + c\gamma x_3^{\beta} & x_3 + c x_3^{\beta} \\ \sigma x_3 + c x_2^{\beta} + c\gamma x_3^{\beta} + c\sigma x_3^{\beta} & x_2 + x_3 + c x_2^{\beta} + c\gamma x_3^{\beta} \end{pmatrix},$$

where $\gamma = \lambda^{\beta} + \lambda$ (in particular γ is a root in $\mathbb{F}_{q'}$ of the polynomial $x^2 + x = \sigma^{\beta} + \sigma$). The linear set associated with the semifield spread $\mathcal{F}_c$ is

$$L_c = \big\{ \big(x + c x^{\beta} + c\gamma y^{\beta}, y + c y^{\beta}, \sigma y + c x^{\beta} + c\gamma y^{\beta} + c\sigma y^{\beta}, \\ x + y + c x^{\beta} + c\gamma y^{\beta}\big) \colon x, y \in \mathbb{F}_{q'} \big\}. \tag{8}$$

Note that also in this case the maximal subfield of linearity of L_c is $\mathbb{F}_q$, i.e. L_c is an $\mathbb{F}_q$-linear set of rank $2n$ of $\mathbb{P}$.

We have

Proposition 3.3. *The $\mathbb{F}_q$-linear sets of $\mathbb{P} = PG(3, q^n)$ of rank $2n$ associated with the* (*pre*)*semifields* (1), (2) *and* (3) *are of scattered type.*

Proof. We prove the statement only for the $\mathbb{F}_q$-linear set

$$L_{K_1} = \big\{ \big(x, y, f y^{\sigma}, x^{\sigma} + g y^{\sigma}\big) \colon x, y \in \mathbb{F}_{q^n} \big\};$$

the other cases are proved with similar arguments. Since the maximum subfield of linearity of L_{K_1} is $\mathbb{F}_q = \mathrm{Fix}(\sigma)$ and for any point $(x, y, f y^{\sigma}, x^{\sigma} + g y^{\sigma})$ of L_{K_1} the point

$(\lambda x, \lambda y, \lambda f y^\sigma, \lambda(x^\sigma + g y^\sigma))$ belongs to L_{K_1} if and only if $\lambda \in \mathbb{F}_q$, then any point of L_{K_1} has weight one, i.e. L_{K_1} is scattered. □

4. Semifields 2-dimensional over the left nucleus and 6-dimensional over the center

We start by proving some geometric properties of an $\mathbb{F}_q$-linear set of $\mathbb{P} = PG(3, q^3)$ of rank 6.

Property 4.1. *Let L be an $\mathbb{F}_q$-linear set of $PG(3, q^3)$ of rank 6 with $\mathbb{F}_q$ as maximal subfield of linearity. A line r of $\mathbb{P}$ is contained in L if and only if $\dim_{\mathbb{F}_q}(r \cap L) \geqslant 4$. Moreover,*

(i) *if $\dim_{\mathbb{F}_q}(r \cap L) > 4$, then L is contained in a plane*;
(ii) *if $\dim_{\mathbb{F}_q}(r \cap L) = 4$, then either there exists a unique point P of r of weight 3 and any other point of r different from P has weight 1 or there exist $q + 1$ points of r of weight 2 and any other point of r has weight 1.*

Proof. Let r be a line of $\mathbb{P}$. If $\dim_{\mathbb{F}_q}(r \cap L) \geqslant 4$, then for any point P of r we have $\dim_{\mathbb{F}_q}(P \cap L) \geqslant 1$; hence r is contained in L. Conversely, suppose that r is contained in L; if $\dim_{\mathbb{F}_q}(r \cap L) \leqslant 3$, then $|r \cap L| \leqslant q^2 + q + 1$; a contradiction. Then $\dim_{\mathbb{F}_q}(r \cap L) \geqslant 4$.

(i) If $\dim_{\mathbb{F}_q}(r \cap L) = 6$, then the maximum subfield of linearity of L is $\mathbb{F}_{q^3}$, a contradiction. Suppose that $\dim_{\mathbb{F}_q}(r \cap L) = 5$, and let P be a point of L not on r. Then the $\mathbb{F}_q$-linear set L is contained in the plane $\langle r, P\rangle$.

(ii) Suppose that $\dim_{\mathbb{F}_q}(r \cap L) = 4$: the proof easily follows from (1) and (2) of Section 2.1. □

Property 4.2. *Let L be an $\mathbb{F}_q$-linear set of $\mathbb{P}$ ($\mathbb{F}_q$ maximal subfield of linearity) contained in a plane π of $\mathbb{P}$. Then either L contains a unique point of weight 3 and it is a union of lines through this point or L is a union of $q^2 + q + 1$ lines and each of them has $q + 1$ points of weight 2.*

Proof. If L has a unique point P of weight 3, then by Property 4.1, any line through P containing another point of L is contained in it. Simply counting arguments show that such lines number either $q^2 + q + 1$ or $q^2 + 1$. Suppose that any point of L has weight at most 2. For any line r of π we get $\dim_{\mathbb{F}_q}(r \cap L) \geqslant 3$. If P is a point of L, an easy computation shows that if P has weight 1, then there exists exactly one line l passing through P such that $\dim_{\mathbb{F}_q}(l \cap L) = 4$; whereas if P has weight 2, then there exist $q + 1$ lines l through P such that $\dim_{\mathbb{F}_q}(l \cap L) = 4$. Hence, by Property 4.1(ii) such lines are contained in L and they contain $q + 1$ points of weight 2. Now the result easily follows. □

We can now prove the following

Theorem 4.3. *If L is an $\mathbb{F}_q$-linear set of rank 6 of $\mathbb{P}$ ($\mathbb{F}_q$ maximal subfield of linearity), then one of the following configurations occurs*:

(0) *L is a union of either $q^2 + q + 1$ or $q^2 + 1$ lines of a pencil of $\mathbb{P}$.*
(1) *L is a union of $q^2 + q + 1$ lines in a plane not belonging to a pencil.*
(2) *L is a union of $q^2 + q + 1$ lines through a point, not all lines in the same plane.*
(3) *L contains a unique point of weight 2, does not contain any line and is not contained in a plane.*

(4) *L contains exactly one line and such a line contains $q+1$ points of weight* 2.
(5) *Any point of L has weight* 1, *i.e. L is a scattered $\mathbb{F}_q$-linear set.*

Proof. Let L be an $\mathbb{F}_q$-linear set of rank 6 of $\mathbb{P}$ having $\mathbb{F}_q$ as maximal subfield of linearity. Suppose that L is contained in a plane π of $\mathbb{P}$; then by Property 4.2 either L contains a unique point of weight 3 and hence Case (0) occurs or L is a union of q^2+q+1 lines of π not belonging to a pencil, each having $q+1$ points of weight 2 and hence Case (1) occurs. If L has a point of weight 3 and L is not contained in a plane we have Case (2). If all points of L have weight 1, then L is not contained in a plane and Case (5) occurs. If L contains exactly one point of weight 2, then by Property 4.1 there are no lines of $\mathbb{P}$ contained in L and hence L is not contained in a plane, so Case (3) occurs. Finally, suppose there are at least two points P and Q of weight 2 and L is not contained in a plane, then $\dim_{\mathbb{F}_q}(r \cap L) = 4$, where $r = \langle P, Q\rangle$; hence the line r is completely contained in L and there exist $q+1$ points of r of weight 2; so Case (4) occurs. □

We say that an $\mathbb{F}_q$-linear set L of rank 6 of $\mathbb{P}$ is *of type* (i) if L has the geometric structure of Theorem 4.3 case (i), $i = 0, \dots, 5$. Accordingly, we say that a semifield 2-dimensional over its left nucleus and 6-dimensional over its center belongs to the family $\mathcal{F}_i$ if the corresponding $\mathbb{F}_q$-linear set of $\mathbb{P}$ is of type (i).

Theorem 4.4. *Semifields belonging to different families $\mathcal{F}_i$ are non-isotopic. Also, if $S \in \mathcal{F}_i$ with $i \in \{0, 3, 4, 5\}$, then $S^\perp \in \mathcal{F}_i$; whereas $S \in \mathcal{F}_1$ if and only if $S^\perp \in \mathcal{F}_2$.*

Proof. Since configurations (i) with $i \in \{0, 3, 4, 5\}$ of Theorem 4.3 are invariant under the action of the subgroup of $P\Gamma O^+(4, q^3)$ fixing the reguli of the hyperbolic quadric $\mathcal{Q}$ of $\mathbb{P}$, from Theorem 1.1 the proof follows. By equality (3) of Subsection 2.1 it turns out that $\dim_{\mathbb{F}_q}(P^\perp \cap L(S^\perp)) = 3 + \dim_{\mathbb{F}_q}(P \cap L(S))$ and $\dim_{\mathbb{F}_q}(r^\perp \cap L(S^\perp)) = \dim_{\mathbb{F}_q}(r \cap L(S))$ for any point P and for any line r of $\mathbb{P}$. From these equalities the second part of the theorem follows. □

The known examples of semifields 2-dimensional over their left nucleus and 6-dimensional over their center, are the following: the semifield associated with the Ganley flock of $PG(3, 3^3)$ (see e.g. [15]); the semifield associated with the Payne–Thas ovoid of $Q(4, 3^3)$; the Generalized Dickson semifields with $\beta = 1$ and $\sigma = q$ or $\beta = q$ and $\sigma = 1$ (see [8, p. 241]); the Knuth semifields $K_1 = (\mathcal{F}, +, \circ)$, with $\mathcal{F} = \mathbb{F}_{q^3} \times \mathbb{F}_{q^3}$ (see (I)); the Knuth semifields $K_2 = (\mathcal{F}, +, \circ)$, with $\mathcal{F} = \mathbb{F}_{q^3} \times \mathbb{F}_{q^3}$ (see (II)); the Generalized twisted fields $\mathcal{F}_c = (\mathcal{F}, +\circ)$, $\mathcal{F} = \mathbb{F}_{q^6}$ (see (III)).

Proposition 4.5. *Let S be a semifield* 2*-dimensional over its left nucleus and* 6*-dimensional over its center $\mathbb{F}_q$.*

(a) *S belongs to the family $\mathcal{F}_0$ if and only if S is isotopic to a Generalized Dickson semifield with $\beta = 1$ and $\sigma = q$ or $\beta = q$ and $\sigma = 1$.*
(b) *S belongs to the family $\mathcal{F}_1$ if and only if $q = 3$ and S is associated with the Payne–Thas ovoid of $Q(4, 3^3)$.*
(c) *S belongs to the family $\mathcal{F}_2$ if and only if $q = 3$ and S is associated with the Ganley flock of $PG(3, 3^3)$.*

Proof. Statement (a) follows from [18, Section 1.5.6]. Concerning (c), if $S \in \mathcal{F}_2$, then S is associated with a flock, 3-dimensional over its nucleus, not of Kantor–Knuth type (for more details see e.g. [15]); then by [4, Corollary 3.2] $q < 14$ and by [3, Theorem 18], S is associated with the Ganley flock of order 27. To prove (b), first recall that by Theorem 4.4, a semifield S belongs to the family $\mathcal{F}_1$ if and only if $S^\perp$ belongs to $\mathcal{F}_2$. So if S belongs to $\mathcal{F}_1$, then it is the translation dual of a semifield associated with the Ganley flock of order 27; i.e. S is associated with the Payne–Thas ovoid of $Q(4,3^3)$. □

In addition to the above examples we have the following

(IV) *Cyclic semifields* $(\mathcal{F},+,\circ)$ of order q^6, 6-dimensional over the center, $\mathcal{F}=\mathbb{F}_{q^6}$ (see [2, p. 415]). Let $V=\mathbb{F}_{q^6}$ and T be an $\mathbb{F}_{q^3}$-linear map of V such that $\alpha T = T\alpha^q$, for any $\alpha \in \mathbb{F}_{q^2}$; then T induces a non-linear collineation of $PG(2,q^2)=PG(V,\mathbb{F}_{q^2})$. Suppose that T does not fix any point of $PG(2,q^2)$ and define the semifield multiplication as follows

$$x \circ y = \big(\alpha_0 + \alpha_1 T + \alpha_2 T^2\big)(x),$$

where $(\alpha_1,\alpha_2,\alpha_3)$ are the components of the element y in the $\mathbb{F}_{q^2}$-basis $\{1, T(1), T^2(1)\}$ of $V=\mathbb{F}_{q^6}$.

In a joint paper with Norman L. Johnson [10] the authors, starting from cyclic semifields with center $\mathbb{F}_q$, construct examples of semifields belonging to the family $\mathcal{F}_4$.

So far, examples of semifields of $\mathbb{P}$ belonging to the family $\mathcal{F}_3$ are not known.

4.1. Semifields of the family $\mathcal{F}_5$

By Theorem 1.1, the study of semifields of the family $\mathcal{F}_5$ up to isotopisms is equivalent to the study of scattered $\mathbb{F}_q$-linear sets of $PG(3,q^3)$ of rank 6 disjoint from the hyperbolic quadric $\mathcal{Q}=Q^+(3,q^3)$ with respect to the action of the subgroup G of $P\Gamma O^+(4,q^3)$ which leaves invariant the reguli of $\mathcal{Q}$. Let $\mathcal{R}$ be the regulus of $\mathcal{Q}$ consisting of the lines $\ell_{\lambda,\mu} = \{(\lambda X_0, \lambda X_1, \mu X_0, \mu X_1)\colon X_0, X_1 \in \mathbb{F}_{q^3}\}$ (where $\lambda,\mu \in \mathbb{F}_{q^3}$) and let $\mathcal{R}'$ be the opposite regulus consisting of the lines $\ell'_{\lambda,\mu} = \{(\lambda X_0, \mu X_0, \lambda X_1, \mu X_1)\colon X_0, X_1 \in \mathbb{F}_{q^3}\}$ (where $\lambda,\mu \in \mathbb{F}_{q^3}$). In what follows, the linear part of G is denoted by $\bar{G}$. An element ϕ of $\bar{G}$ is defined by a matrix $A = B'B$, where

$$B' = \begin{pmatrix} a' & c' & 0 & 0 \\ b' & d' & 0 & 0 \\ 0 & 0 & a' & c' \\ 0 & 0 & b' & d' \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & c \\ b & 0 & d & 0 \\ 0 & b & 0 & d \end{pmatrix}, \tag{9}$$

with $a'd' - c'b' \neq 0$ and $ad - cb \neq 0$, i.e. $\phi = \theta'\theta$ where θ' and θ are the linear collineations of G defined by the matrices B' and B, respectively. The collineations of type θ' (respectively θ) of $\bar{G}$ fix the regulus $\mathcal{R}$ (respectively $\mathcal{R}'$) linewise and they form a subgroup of $\bar{G}$ acting on the regulus $\mathcal{R}'$ (respectively $\mathcal{R}$) as the linear group $PGL(2,q^3)$.

As proved in Subsection 2.2, to any semifield of the family $\mathcal{F}_5$ there corresponds an $\mathbb{F}_q$-pseudoregulus $\mathcal{L}(S)$ of $\mathbb{P}=PG(3,q^3)$. Moreover, semifields of $\mathcal{F}_5$ whose associated $\mathbb{F}_q$-pseudoreguli of $\mathbb{P}$ have different "positions" with respect to the quadric $\mathcal{Q}$ are not isotopic, as we prove in the following

Proposition 4.6. *If S and S' are two isotopic semifields of the family $\mathcal{F}_5$, then the associated $\mathbb{F}_q$-pseudoreguli of $\mathbb{P}$ are isomorphic under the action of the group G.*

Proof. Let $L(S)$ and $L(S')$ be the $\mathbb{F}_q$-linear sets of $\mathbb{P}$ associated with S and S', respectively. Since S and S' are isotopic, there exists an element ϕ of G such that $\phi(L(S)) = L(S')$ (Theorem 1.1). As proved in Section 2.2, the $\mathbb{F}_q$-pseudoreguli $\mathcal{L}(S)$ and $\mathcal{L}(S')$ consist of all (q^2+q+1)-secant lines of $L(S)$ and $L(S')$, respectively. Then $\phi(\mathcal{L}(S)) = \mathcal{L}(S')$ and, in particular, the pair of transversal lines of $\mathcal{L}(S)$ is mapped under ϕ in the pair of transversal lines of $\mathcal{L}(S')$. □

We conclude this subsection by giving a description of the $\mathbb{F}_q$-pseudoreguli associated with the known examples of semifields of $\mathcal{F}_5$.

(A) *$\mathbb{F}_q$-pseudoreguli associated with the Knuth semifields K_1 and K_2*

Let K_1 and K_2 be the Knuth semifields of $\mathcal{F}_5$ and let $L(K_1)$ and $L(K_2)$ be the corresponding $\mathbb{F}_q$-linear sets of $\mathbb{P} = PG(3,q^3)$ (see (4) and (5) with $1 \neq \sigma \in \mathrm{Aut}(\mathbb{F}_{q^3} \mid \mathbb{F}_q)$). Direct computations show that the $\mathbb{F}_q$-pseudoreguli of $\mathbb{P}$ associated with $L(K_1)$ and $L(K_2)$ consist of the lines

$$r_\alpha\colon\quad \begin{cases} x_1 = \alpha x_0, \\ (1+\alpha^\sigma g)x_2 = \alpha^\sigma f x_3, \end{cases} \quad \alpha \in \mathbb{F}_{q^3} \quad \text{and} \quad r_\infty\colon\quad \begin{cases} x_0 = 0, \\ g x_2 = f x_3 \end{cases}$$

and

$$r_\alpha\colon\quad \begin{cases} x_2 = \alpha^{\sigma^{-1}} f x_0, \\ \alpha x_3 = (1+\alpha g)x_1, \end{cases} \quad \alpha \in \mathbb{F}_{q^3} \quad \text{and} \quad r_\infty\colon\quad \begin{cases} x_0 = 0, \\ x_3 = g x_1, \end{cases}$$

respectively.

The transversal lines ℓ and ℓ' of $\mathcal{L}(K_1)$ and m and m' of $\mathcal{L}(K_2)$ have equations

$$l\colon\quad x_2 = x_3 = 0, \qquad l'\colon\quad x_0 = x_1 = 0,$$
$$m\colon\quad x_0 = x_2 = 0, \qquad m'\colon\quad x_1 = x_3 = 0,$$

and hence they belong to the reguli $\mathcal{R}$ and $\mathcal{R}'$, respectively. Then, we have the following

Property 4.7. *The transversal lines of $\mathcal{L}(K_1)$ and $\mathcal{L}(K_2)$ are contained in the quadric $\mathcal{Q}$ and they belong to the reguli $\mathcal{R}$ and $\mathcal{R}'$, respectively.*

(B) *$\mathbb{F}_q$-pseudoreguli associated with the generalized twisted fields*

Let $\mathcal{F}_c$ be a generalized twisted field belonging to the family $\mathcal{F}_5$ and let q be odd. The $\mathbb{F}_q$-pseudoregulus $\mathcal{L}(\mathcal{F}_c)$ associated with $L(\mathcal{F}_c)$ (see (7) with $1 \neq \beta \in \mathrm{Aut}(\mathbb{F}_{q^6} \mid \mathbb{F}_q)$) consists of the lines

$$r_\alpha = \bigl\langle (\alpha, 1, \sigma, \alpha), \bigl(c_1\alpha^\beta + c_2 a\sigma, c_2\alpha^\beta + c_1 a, -\sigma\bigl(c_2\alpha^\beta + ac_1\bigr), -\bigl(c_1\alpha^\beta + c_2 a\sigma\bigr)\bigr)\bigr\rangle,$$

where $\alpha \in \mathbb{F}_{q^3}$, and

$$r_\infty = \bigl\langle (1,0,0,1), (c_1, c_2, -\sigma c_2, -c_1)\bigr\rangle.$$

A direct computation shows that

$$t_c = \bigl\{(x, y, \sigma y, x)\colon\ x, y \in \mathbb{F}_{q^3}\bigr\}$$

and

$$t'_c = \bigl\{(x, y, -\sigma y, -x)\colon\ x, y \in \mathbb{F}_{q^3}\bigr\}$$

are the transversal lines of $\mathcal{L}(\mathcal{F}_c)$. Also $t'_c = t_c^{\perp}$. Moreover, $r_0^{\perp} = r_\infty$ and, for any $\alpha \in \mathbb{F}^*_{q^3}$, $r_\alpha^{\perp} = r_{\alpha'}$, where $\alpha' = \frac{\sigma}{\alpha}$.

Let $\mathcal{F}_c$ be a generalized twisted field belonging to the family $\mathcal{F}_5$ and let q be even. The $\mathbb{F}_q$-pseudoregulus $\mathcal{L}(\mathcal{F}_c)$ associated with $L(\mathcal{F}_c)$ (see (8) with $\beta \neq 1$ an $\mathbb{F}_q$-automorphism of $\mathbb{F}_{q^6}$) consists of the lines

$$r_\alpha = \big\langle (\alpha, 1, \sigma, \alpha+1), \big(\alpha^\beta + \gamma, 1, \alpha^\beta + \gamma + \sigma, \alpha^\beta + \gamma\big)\big\rangle,$$

where $\alpha \in \mathbb{F}_{q^3}$ and

$$r_\infty = \big\langle (1,0,0,1), (1,0,1,1)\big\rangle.$$

Also, $t_c = \{(x, y, \sigma y, x+y)\colon x, y \in \mathbb{F}_{q^3}\}$ and $t'_c = \{(x, y, x+\sigma y, x)\colon x, y \in \mathbb{F}_{q^3}\}$ are the transversal lines of $\mathcal{L}(\mathcal{F}_c)$ and $t'_c = t_c^{\perp}$. Obviously, in this case any line r_α is a tangent line to $\mathcal{Q}$. Hence, in both cases (q odd and q even) we have the following

Property 4.8. *The transversal lines t_c and t'_c of $\mathcal{L}(\mathcal{F}_c)$ are pairwise polar external lines of $\mathcal{Q}$. Also, the set of lines of $\mathcal{L}(\mathcal{F}_c)$ is preserved by the polarity $\perp$ induced by $\mathcal{Q}$.*

Moreover, we prove the next

Proposition 4.9. *Let q be even and let t be a line of $\mathbb{P} = PG(3, q^3)$, which is external to $\mathcal{Q}$. The stabilizer $\bar{G}_t$ of t in $\bar{G}$ acts transitively on the transversal lines of t and $t^{\perp}$.*

Proof. Since $\bar{G}$ acts transitively on the set of external lines to $\mathcal{Q}$, we can suppose $t = \{(x, y, \sigma y, x + y)\colon x, y \in \mathbb{F}_{q^3}\}$. The stabilizer $\bar{G}_t$ consists of the elements ϕ defined by matrices $A = BB'$ (see (9)) with $a' = d' + (1-\delta)b'$, $c' = b'\sigma + \delta d'$, $b = c\sigma + \delta a$, $d = a + (1-\delta)c$, where $\delta = 0, 1$ and (a, c), $(b', d') \neq (0, 0)$. An easy computation shows that $\bar{G}_t$ acts transitively on t. Hence, we can fix $M = (1, 0, 0, 1) \in t$, and an element of $\bar{G}_t$ belongs to $\bar{G}_{t,M}$ if $ab' + cd' = 0$. Fix the point $R = (0, 1, \sigma, 0) \in t^{\perp}$ and let $O(R)$ be the orbit of R under the action of the group $\bar{G}_{t,M}$. The points $R^\phi \in O(R)$, where $\phi \in \bar{G}_{t,M}$ with $\delta = 0$ are $\{(1, \frac{\sigma c^2 + a^2}{\sigma c^2}, 1 + \sigma(\frac{\sigma c^2+a^2}{\sigma c^2}), 1)\} \cup \{R\}$ with $a, c \in \mathbb{F}_{q^3}$, $c \neq 0$. Since any element of $\mathbb{F}_{q^3}$ can be written as $\frac{\sigma c^2+a^2}{\sigma c^2}$ for some elements $a, c \in \mathbb{F}_{q^3}$, it follows $O(R) = t^{\perp}$. This implies that $\bar{G}_t$ acts transitively on the transversal lines of t and $t^{\perp}$. $\square$

Let t be an external line to $\mathcal{Q}$ and let r be a transversal line to t and $t^{\perp}$. The stabilizer $\bar{G}_{t,r}$ of the lines t and r in $\bar{G}$ has order 2. Denote by ϕ_r the non-trivial element of $\bar{G}_{t,r}$, if m is a transversal line to t and $t^{\perp}$ with $m \cap r = \emptyset$, the line m^{ϕ_r} is said *conjugate* of m with respect to t and r (or simply conjugate of m with respect to r). For instance, if $t = t_c$ and r is the line r_∞, then the non-trivial element of $\bar{G}_{t_c,r_\infty}$ is $\phi_{r_\infty}(x_0, x_1, x_2, x_3) = (x_0 + x_1, x_1, x_0 + x_1 + x_2 + x_3, x_1 + x_3)$.

Proposition 4.10. *Let $\mathcal{L}(\mathcal{F}_c)$ be the $\mathbb{F}_q$-pseudoregulus associated with a generalized twisted field $\mathcal{F}_c$, $c \in \mathbb{F}_{q^3}$, with q even, and let t_c and $t_c^{\perp}$ be the two transversal lines of $\mathcal{L}(\mathcal{F}_c)$. Then, for any line r of $\mathcal{L}(\mathcal{F}_c)$, the stabilizer $\bar{G}_{t_c,r}$ leaves $\mathcal{L}(\mathcal{F}_c)$ invariant.*

Proof. First, consider the line r_∞. For any line r_k, with $k \in \mathbb{F}_{q^3}$, the image of r_k under the action of the group $\bar{G}_{t_c,r_\infty}$ is r_{k+1}. Now, for any $\alpha \in \mathbb{F}_{q^3}$, consider the line $r_\alpha = \langle P_\alpha, P'_\alpha\rangle$, where

$P_\alpha = (\alpha, 1, \sigma, \alpha + 1)$ and $P'_\alpha = (\alpha^\beta + \gamma, 1, \alpha^\beta + \gamma + \sigma, \alpha^\beta + \gamma)$. Straightforward calculations show that the image of r_∞ under $\bar{G}_{t_c,r_\alpha}$ is $r_{\alpha^2+\sigma}$, whereas for any line r_h, with $h \in \mathbb{F}_{q^3}$ and $h \neq \alpha$, the image of r_h under $\bar{G}_{t_c,r_\alpha}$ is r_ξ, where $\xi = \frac{\alpha^2 h + h\sigma + \alpha^2}{\alpha^2 + h + \sigma}$. Hence, the stabilizer $\bar{G}_{t_c,r_\alpha}$ leaves $\mathcal{L}(\mathcal{F}_c)$ invariant. □

In the following we will show that the geometric configurations of the $\mathbb{F}_q$-pseudoreguli associated with Knuth semifields and generalized twisted fields, described in this subsection, characterize them.

4.2. A geometric characterization of the known classes of semifields of $\mathcal{F}_5$

Let $\mathcal{Q} = Q^+(3, q^3)$: $X_0X_3 - X_1X_2 = 0$ be as in the previous section and recall that the two reguli of $\mathcal{Q}$ are $\mathcal{R}' = \{l'_{\lambda,\mu}\colon \lambda, \mu \in \mathbb{F}_{q^3}\}$, $l'_{\lambda,\mu} = \{(\lambda X_0, \mu X_0, \lambda X_2, \mu X_2)\colon X_0, X_2 \in \mathbb{F}_{q^3}\}$ and $\mathcal{R} = \{l_{\lambda,\mu}\colon \lambda, \mu \in \mathbb{F}_{q^3}\}$, $l_{\lambda,\mu} = \{(\lambda X_0, \lambda X_1, \mu X_0, \mu X_1)\colon X_0, X_1 \in \mathbb{F}_{q^3}\}$. We have the following

Theorem 4.11. *Let S be a semifield in $\mathcal{F}_5$. Then S is isotopic to a Knuth semifield K_1 (respectively K_2) if and only if the transversal lines of $\mathcal{L}(S)$ are contained in the regulus $\mathcal{R}$ (respectively $\mathcal{R}'$) of $\mathcal{Q}$.*

Proof. Let S be a semifield of $\mathcal{F}_5$ such that the two transversal lines, t and t', of the associated $\mathbb{F}_q$-pseudoregulus $\mathcal{L}(S)$ of $\mathbb{P} = PG(3, q^3)$ belong to the regulus $\mathcal{R}$. Since the group G acts 2-transitively on the lines of $\mathcal{R}$, we can suppose that $t = \{(x_0, x_1, 0, 0)\colon x_0, x_1 \in \mathbb{F}_{q^3}\}$ and $t' = \{(0, 0, x_2, x_3)\colon x_2, x_3 \in \mathbb{F}_{q^3}\}$. Note that the stabilizer $H = G_{\{t,t'\}}$ in the group G of the lines t and t' acts transitively on the points of t. If P is any point of t, then the stabilizer H_P of P in H fixes the point $P^\perp \cap t'$ and acts transitively on the remaining points of t'. This means that we can suppose, without loss of generality, that the line r with equations $X_1 = X_2 = 0$ belongs to $\mathcal{L}(S)$. Let $R = t \cap r = (1, 0, 0, 0)$ and $R' = t' \cap r = (0, 0, 0, 1)$. Embed $\mathbb{P} = PG(3, q^3)$ in a 5-dimensional projective space $\Sigma' = PG(5, q^3)$ in such a way that $\mathbb{P}$ has equations $X_4 = X_5 = 0$. The $\mathbb{F}_q$-linear set $L(S)$ is obtained as the projection of a canonical subgeometry $\Sigma \simeq PG(5, q)$ of Σ' from a line l onto $\mathbb{P}$. Let ω be a semilinear collineation of Σ' of order 3 such that $\Sigma = \mathrm{Fix}(\omega)$. Since $L(S)$ does not depend on the choice of the axis $\mathbb{P}$ (see Remark 2.4) we can suppose that $t = l^\omega$ and $t' = l^{\omega^2}$, and hence $R^\omega = R'$. We can choose homogeneous projective coordinates in Σ' in such a way that the line l has equations $X_0 = X_1 = X_2 = X_3 = 0$. Moreover, since the pointwise stabilizer of $\mathbb{P}$ in $PGL(6, q^3)$ acts 3-transitively on the line l, we can suppose that $R^{\omega^2} = (1, 0, 0, 0, 0, 0)^{\omega^2} = (0, 0, 0, 0, 1, 0)$, $(0, 1, 0, 0, 0, 0)^{\omega^2} = (0, 0, 0, 0, 0, 1)$ and $(1, 1, 0, 0, 0, 0)^{\omega^2} = (0, 0, 0, 0, 1, 1)$. Let $P = (0, 0, 0, 0, 1, 0)$, $Q = (0, 0, 0, 0, 0, 1)$ and $M = (0, 0, 0, 0, 1, 1)$. Since $\mathrm{Fix}(\omega) = \mathrm{Fix}(\omega^2) = \Sigma$ and since one of these semilinear collineations has associated automorphism $x \to x^q$, we can suppose that this is ω. Hence, let $\omega\colon \underline{x} = (x_0, x_1, x_2, x_3, x_4, x_5) \in \Sigma' \to A\underline{x}^q \in \Sigma'$ where $A = (a_{ij})$, $i, j \in \{0, 1, 2, 3, 4, 5\}$ is a non-singular 6×6 matrix over $\mathbb{F}_{q^3}$. By requiring that $P^\omega = (1, 0, 0, 0, 0, 0)$, $P^{\omega^2} = (0, 0, 0, 1, 0, 0)$ and $P^{\omega^3} = P$ we get: $a_{14} = a_{24} = a_{34} = a_{44} = a_{54} = 0$, $a_{00} = a_{10} = a_{20} = a_{40} = a_{50} = 0$ and $a_{03} = a_{13} = a_{23} = a_{33} = a_{53} = 0$, respectively. Also, by requiring $Q^\omega = (0, 1, 0, 0, 0, 0)$, $Q^{\omega^2} \in l^{\omega^2} \setminus \{P^{\omega^2}\}$ and $Q^{\omega^3} = Q$ we get $a_{05} = a_{25} = a_{35} = a_{45} = a_{55} = 0$, $a_{01} = a_{11} = a_{41} = a_{51} = 0$ and $a_{02} = a_{12} = a_{22} = 0$, $a_{21}^q a_{42} = -a_{43} a_{31}^q$. Finally, from $M^\omega = (1, 1, 0, 0, 0, 0)$, $M^{\omega^2} \in l^{\omega^2} \setminus \{P^{\omega^2}\}$ and $M^{\omega^3} = M$ it follows $a_{04} = a_{15}$ and $a_{30}^q a_{43} = a_{21}^q a_{52}$. Since $a_{30} \neq 0$,

$a_{21} \neq 0$ and $a_{04} \neq 0$, we can put $a_{30} = 1$ and from $N^{\omega^3} = N$, where $N = (1,1,1,1,1,1)$ we get $a_{52} = \frac{\alpha}{a_{04}^{q^2} a_{21}^{q}}$ with $\alpha \in \mathbb{F}_q$. Then the matrix A has the following form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a_{04} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{04} \\ 0 & a_{21} & 0 & 0 & 0 & 0 \\ 1 & a_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{42} & a_{43} & 0 & 0 \\ 0 & 0 & a_{52} & 0 & 0 & 0 \end{pmatrix},$$

where $a_{42} = \frac{-\alpha a_{31}^q}{a_{21}^q a_{04}^{q^2}}$, $a_{43} = \frac{\alpha}{a_{04}^{q^2}}$ and $a_{52} = \frac{\alpha}{a_{04}^{q^2} a_{21}^{q}}$. A direct computation now shows that

$$\Sigma = \mathrm{Fix}(\omega) = \left\{ \left(x, y, \frac{a_{21}}{\rho} y^q, \frac{x^q + a_{31} y^q}{\rho}, \frac{\rho^{q^2}}{a_{04}^{q^2}} x^{q^2}, \frac{\rho^{q^2}}{a_{04}^{q^2}} y^{q^2} \right) \colon x, y \in \mathbb{F}_{q^3} \right\},$$

where $\rho \in \mathbb{F}_{q^3}^*$ and $\rho^{q^2+q+1} = \alpha$.

By projecting Σ from the line l onto $\mathbb{P}$ we get the $\mathbb{F}_q$-linear set

$$L(S) = \left\{ \left(x, y, \frac{a_{21}}{\rho} y^q, \frac{x^q}{\rho} + \frac{a_{31}}{\rho} y^q \right) \colon x, y \in \mathbb{F}_{q^3} \right\},$$

which is isomorphic, under the collineation $\varphi : (X_0, X_1, X_2, X_3) \to (X_0, X_1, \rho X_2, \rho X_3)$ belonging to the group G, to the $\mathbb{F}_q$-linear set $L(K_1) = \{(x, y, a_{21} y^q, x^q + a_{31} y^q) \colon x, y \in \mathbb{F}_{q^3}\}$ where K_1 is a Knuth semifield (I) (see Section 3.1), with $f = a_{21}$, $g = a_{31}$ and $\sigma = q$.

Suppose, on the other hand, that the $\mathbb{F}_q$-pseudoregulus associated to the semifield S in $\mathcal{F}_5$ has transversal lines t and t' belonging to the opposite ruling $\mathcal{R}'$ of $\mathcal{Q}$. Consider the collineation ϕ of $\mathbb{P}$ defined by $\phi : (x_0, x_1, x_2, x_3) \to (x_0, x_2, x_1, x_3)$; this collineation fixes $\mathcal{Q}$, interchanges the reguli and maps any point $P = \langle M \rangle_{\mathbb{F}_q^3}$ of the $\mathbb{F}_q$-linear set $L(S)$ to the point $P' = \langle M^t \rangle_{\mathbb{F}_q^3}$ of $\mathbb{P}$, where M^t is the transpose matrix of M. Hence by applying ϕ to $L(S)$, we obtain an $\mathbb{F}_q$-linear set $\phi(L(S)) = L(S')$ of rank 6 of $\mathbb{P}$ lying on an $\mathbb{F}_q$-pseudoregulus whose transversals belong to the regulus $\mathcal{R}$. By the first part of the proof, S' is isotopic to a Knuth semifield K_1 and hence S is isotopic to the transpose of a Knuth semifield K_1, i.e. S is isotopic to a Knuth semifield K_2 (see Remark 3.2). $\square$

Theorem 4.12. *Let S be a semifield in $\mathcal{F}_5$, q odd. Then S is isomorphic to a generalized twisted field if and only if* (i) *the transversal lines t and t' of $\mathcal{L}(S)$ are external to $\mathcal{Q}$ and they are pairwise polar*; (ii) *the set of lines of $\mathcal{L}(S)$ is invariant under $\perp$.*

Proof. Let S be a semifield of $\mathcal{F}_5$, q odd, such that the associated $\mathbb{F}_q$-pseudoregulus $\mathcal{L}(S)$ satisfies conditions (i) and (ii). Since the group G acts transitively on the external lines to $\mathcal{Q}$ we can suppose that $t = \{(x_0, x_1, \sigma x_1, x_0) \colon x_0, x_1 \in \mathbb{F}_{q^3}\}$, where σ is a fixed non-square element in $\mathbb{F}_{q^3}$; hence $t^\perp = \{(x_0, x_1, -\sigma x_1, -x_0) \colon x_0, x_1 \in \mathbb{F}_{q^3}\}$. Embed $\mathbb{P}$ in $\Sigma' \simeq PG(5, q^3)$; then the $\mathbb{F}_q$-linear set $L(S)$ is obtained as the projection of a canonical subgeometry $\Sigma \simeq PG(5, q)$ of Σ' from a line l disjoint from $\mathbb{P}$. Without loss of generality, we can suppose that $\mathbb{P}$ has equations $X_4 = X_5 = 0$ and that l has equations $X_0 = X_1 = X_2 = X_3 = 0$. Let ω be a semilinear collineation of Σ' of order 3 such that $\Sigma = \mathrm{Fix}(\omega)$. Since $L(S)$ does not depend on the choice of the axis $\mathbb{P}$, we can suppose that $t = l^\omega$ and $t^\perp = l^{\omega^2}$. Moreover, if $T = (0,0,0,0,1,0)$, $R = (0,0,0,0,0,1)$ and $S = (0,0,0,0,1,1)$, as in the proof of Theorem 4.11, we can suppose that $T^\omega = M = (1,0,0,1,0,0)$, $R^\omega = N = (0,1,\sigma,0,0,0)$ and $S^\omega = C = (1,1,\sigma,1,0,0)$. Again, we can suppose that ω has associated automorphism $x \to x^q$. Let $\omega : \underline{x} = (x_0,x_1,x_2,x_3,x_4,x_5) \subset \Sigma' \to A\underline{x}^q \subset \Sigma'$, where $A = (a_{ij})$, $i,j \in \{0,1,2,3,4,5\}$ is a non-singular matrix over $\mathbb{F}_{q^3}$. By requiring that $T^\omega = M$, $R^\omega = N$ and $S^\omega = C$ we get the following conditions $a_{14} = a_{24} = a_{44} = a_{54} = 0$, $a_{04} = a_{34}$, $a_{05} = a_{35} = a_{45} = a_{55} = 0$, $a_{25} = \sigma a_{15}$ and $a_{04} = a_{15}$. By requiring that $t^\omega = t^\perp$ and $(t^\perp)^\omega = l$ we have $a_{00} = -a_{30}$, $a_{02} = -a_{32}$, $a_{20} = -\sigma a_{10}$ and $a_{22} = -\sigma a_{12}$. Finally, if $\{P_0,P_1,P_2,P_3,P_4,P_5,P\}$ is the standard frame of Σ', from $P_i^{\omega^3} = P_i$ $(i = 0,\dots,5)$ and from $P^{\omega^3} = P$, we get

$$a_{50}a_{00}^q - \sigma^q a_{52}a_{10}^q = 0, \tag{10}$$

$$a_{10}a_{40}^{q^2} + \sigma^q a_{12}a_{50}^{q^2} = 0,$$

$$a_{04}\big(a_{40}^q a_{00}^{q^2} - \sigma^{q^2} a_{42}^q a_{10}^{q^2}\big) = a_{00}a_{04}^q a_{40}^{q^2} + \sigma^q a_{02}a_{04}^q a_{50}^{q^2}, \tag{11}$$

$$a_{40}a_{02}^q - \sigma^q a_{42}a_{12}^q = 0, \tag{12}$$

$$a_{00}a_{42}^{q^2} + \sigma^q a_{02}a_{52}^{q^2} = 0,$$

$$a_{10}a_{04}^q a_{42}^{q^2} + \sigma^q a_{12}a_{04}^q a_{52}^{q^2} = -a_{04}\big(a_{50}^q a_{02}^{q^2} - \sigma^{q^2} a_{52}^q a_{12}^{q^2}\big),$$

and

$$\begin{aligned}
&a_{00}a_{04}^q a_{40}^{q^2} + \sigma^q a_{02}a_{04}^q a_{50}^{q^2} + a_{04}\big(a_{40}^q a_{00}^{q^2} - \sigma^{q^2} a_{42}^q a_{10}^{q^2}\big)\\
&\quad = -\sigma a_{10}a_{04}^q a_{42}^{q^2} - \sigma^{q+1} a_{12}a_{04}^q a_{52}^{q^2} + \sigma a_{04}\big(a_{50}^q a_{02}^{q^2} - \sigma^{q^2} a_{52}^q a_{12}^{q^2}\big)\\
&\quad = 2a_{04}^{q^2}\big(a_{40}a_{00}^q - \sigma^q a_{42}a_{10}^q\big) = 2\sigma^{q^2} a_{04}^{q^2}\big(a_{50}a_{02}^q - \sigma^q a_{52}a_{12}^q\big).
\end{aligned}$$

Let

$$\begin{aligned}
A &= a_{00}a_{04}^q a_{40}^{q^2} + \sigma^q a_{02}a_{04}^q a_{50}^{q^2} + a_{04}\big(a_{40}^q a_{00}^{q^2} - \sigma^{q^2} a_{42}^q a_{10}^{q^2}\big),\\
B &= -\sigma a_{10}a_{04}^q a_{42}^{q^2} - \sigma^{q+1} a_{12}a_{04}^q a_{52}^{q^2} + \sigma a_{04}\big(a_{50}^q a_{02}^{q^2} - \sigma^{q^2} a_{52}^q a_{12}^{q^2}\big),\\
C &= 2a_{04}^{q^2}\big(a_{40}a_{00}^q - \sigma^q a_{42}a_{10}^q\big),\\
D &= 2\sigma^{q^2} a_{04}^{q^2}\big(a_{50}a_{02}^q - \sigma^q a_{52}a_{12}^q\big).
\end{aligned}$$

By Eq. (11) and $A = C$ we get

$$\gamma = a_{04}\big(a_{40}^q a_{00}^{q^2} - \sigma^{q^2} a_{42}^q a_{10}^{q^2}\big) \in \mathbb{F}_q$$

and hence

$$A = B = C = D = 2\gamma.$$

From $B = D$, it follows

$$\gamma = -\sigma\big(a_{10}a_{04}^q a_{42}^{q^2} + \sigma^q a_{12}a_{04}^q a_{52}^{q^2}\big).$$

By using Eq. (10) and $D = 2\gamma$ we obtain

$$a_{50} = \frac{a_{10}^q \gamma}{\sigma^{q^2} a_{04}^{q^2} \Delta} \quad \text{and} \quad a_{52} = \frac{a_{00}^q \gamma}{\sigma^{q^2+q} a_{04}^{q^2} \Delta},$$

where $\Delta = a_{10}^q a_{02}^q - a_{00}^q a_{12}^q$ and, by using Eq. (12) and $C = 2\gamma$ we have

$$a_{40} = \frac{-a_{12}^q \gamma}{a_{04}^{q^2} \Delta} \quad \text{and} \quad a_{42} = \frac{-a_{02}^q \gamma}{\sigma^q a_{04}^{q^2} \Delta}.$$

Now, let $r = \langle Q, Q^\omega \rangle$ be a line of $\mathcal{L}(S)$ with $Q \in t$ and $Q^\omega \in t^\perp$. By the assumption (ii), the line $r^\perp$ belongs to $\mathcal{L}(S)$ and hence $(Q^\omega)^\perp \cap t^\perp = (Q^\perp \cap t)^\omega$. From these conditions we get

$$a_{02} = \xi \sigma a_{10}, \qquad a_{12} = \xi a_{00}, \qquad a_{40} = \frac{\gamma a_{00}^q}{\alpha a_{04}^{q^2}},$$

$$a_{42} = \frac{\gamma a_{10}^q}{\alpha a_{04}^{q^2}}, \qquad a_{50} = \frac{-\gamma a_{10}^q}{\alpha \sigma^{q^2} \xi^q a_{04}^{q^2}}, \qquad a_{52} = \frac{-\gamma a_{00}^q}{\alpha \sigma^{q^2+q} \xi^q a_{04}^{q^2}},$$

where $\alpha = a_{00}^{2q} - \sigma^q a_{10}^{2q}$ and $\xi^2 = \frac{1}{\sigma^{q+1}}$. Let $a \in \mathbb{F}_{q^3}$ such that $a^2 = \sigma^{q-1}$, hence $\xi^2 = \frac{1}{\sigma^2 a^2}$. By choosing $\xi = \frac{1}{\sigma a}$ and setting $a_{04} = 2$, the matrix A becomes

$$\begin{pmatrix} a_{00} & a\sigma a_{10} & \frac{a_{10}}{a} & a_{00} & 2 & 0 \\ a_{10} & a a_{00} & \frac{a_{00}}{a\sigma} & a_{10} & 0 & 2 \\ -\sigma a_{10} & -a\sigma a_{00} & -\frac{a_{00}}{a} & -\sigma a_{10} & 0 & 2\sigma \\ -a_{00} & -a\sigma a_{10} & -\frac{a_{10}}{a} & -a_{00} & 2 & 0 \\ a_{40} & -\sigma^q a_{42} & a_{42} & -a_{40} & 0 & 0 \\ -\frac{a_{42}}{a^q} & \frac{a_{40}}{\alpha a^q} & -\frac{a_{40}}{\sigma^q a^q} & \frac{a_{42}}{a^q} & 0 & 0 \end{pmatrix},$$

where $a_{40} = \frac{\gamma a_{00}^q}{2\alpha}$ and $a_{42} = \frac{\gamma a_{10}^q}{2\alpha}$. Direct computation shows that

$$\begin{aligned}\Sigma = \mathrm{Fix}(\omega) = \Bigg\{ \Bigg(& 2a_{00}x^q + \rho^q x + 2a\sigma a_{10} y^q, 2a_{10}x^q + 2aa_{00}y^q + \rho^q y, \\ & -2\sigma a_{10}x^q + \sigma\rho^q y - 2a\sigma a_{00} y^q, \rho^q x - 2a_{00}x^q - 2a\sigma a_{10} y^q, \\ & \frac{\rho^{q+1}}{2} x^{q^2}, \frac{\rho^{q+1}}{2} y^{q^2} \Bigg) \colon x, y \in \mathbb{F}_{q^3} \Bigg\},\end{aligned}$$

where ρ is an element of $\mathbb{F}_{q^3}^*$ such that $\rho^{q^2+q+1} = 4\gamma$. Finally by projecting Σ from the line l onto $\mathbb{P}$, we get the $\mathbb{F}_q$-linear set

$$\begin{aligned}L(S) = \Bigg\{ \Bigg(& x + \frac{2a_{00}}{\rho^q} x^q + 2a_{10} \frac{a\sigma}{\rho^q} y^q, y + \frac{2a_{10}}{\rho^q} x^q + \frac{2aa_{00}}{\rho^q} y^q, \\ & \sigma y - \frac{2\sigma a_{10}}{\rho^q} x^q - \frac{2a\sigma a_{00}}{\rho^q} y^q, x - \frac{2a_{00}}{\rho^q} x^q - \frac{2a\sigma a_{10}}{\rho^q} y^q \Bigg) \colon x, y \in \mathbb{F}_{q^3} \Bigg\},\end{aligned}$$

which is the $\mathbb{F}_q$-linear set associated with the generalized twisted field $\mathcal{F}_c$ (q odd) with $c = -\frac{2a_{00}}{\rho^q} - \frac{2a_{10}}{\rho^q}\lambda$ and $\beta = q$. Finally, if $\xi = -\frac{1}{\sigma a}$, using the same arguments as in the previous case, we get the $\mathbb{F}_q$-linear set associated with the generalized twisted field $\mathcal{F}_c$ (q odd), with $c = -\frac{2a_{00}}{\rho^q} + \frac{2a_{10}}{\rho^q}\lambda$ and $\beta = q^4$. □

Theorem 4.13. *Let S be a semifield of $\mathcal{F}_5$, q even. Then S is isotopic to a generalized twisted field if and only if* (i) *the transversal lines t and t' of $\mathcal{L}(S)$ are external to $\mathcal{Q}$ and they are pairwise polar*; (ii) *for any line r of $\mathcal{L}(S)$ the stabilizer $\bar{G}_{t,r}$ leaves $\mathcal{L}(S)$ invariant.*

Proof. Let S be a semifield of $\mathcal{F}_5$, q even, such that the associated $\mathbb{F}_q$-pseudoregulus $\mathcal{L}(S)$ satisfies conditions (i) and (ii). Since the group G acts transitively on the external lines to $\mathcal{Q}$, we can suppose that $t = \{(x_0, x_1, \sigma x_1, x_0 + x_1)\colon\ x_0, x_1 \in \mathbb{F}_{q^3}\}$ where σ is an element of $\mathbb{F}_{q^3}$ such that the polynomial $x^2 + x + \sigma$ has no roots in $\mathbb{F}_{q^3}$. Hence $t' = t^\perp = \{(x_0, x_1, x_0 + \sigma x_1, x_0)\colon$ $x_0, x_1 \in \mathbb{F}_{q^3}\}$. Note that the stabilizer $\bar{G}_t$ of the line t in $\bar{G}$ acts transitively on the points of t. Let M be any point of t; then $\bar{G}_{\{t,M\}}$ (the stabilizer of the point M in $\bar{G}_t$) acts transitively on $t^\perp$ (see Proposition 4.9). This means that we can suppose, without loss of generality, that the line $\langle M, M'\rangle$, where $M = (1, 0, 0, 1)$ and $M' = (1, 0, 1, 1)$, is contained in $\mathcal{L}(S)$.

Embed $\mathbb{P}$ in $\Sigma' = PG(5, q^3)$; then the $\mathbb{F}_q$-linear set $L(S)$ is obtained as the projection of a canonical subgeometry $\Sigma \simeq PG(5, q)$ of Σ' from a line l disjoint from $\mathbb{P}$ onto $\mathbb{P}$. Let ω be a semilinear collineation of Σ' of order 3 such that $\Sigma = \mathrm{Fix}(\omega)$. Since $L(S)$ does not depend on the choice of the axis, we can suppose that $t = l^\omega$ and $t' = l^{\omega^2}$. If $T = (0, 0, 0, 0, 1, 0)$, $R = (0, 0, 0, 0, 0, 1)$ and $S = (0, 0, 0, 0, 1, 1)$, as in the previous proof, we can suppose that $T^\omega = M = (1, 0, 0, 1, 0, 0)$, $R^\omega = N = (0, 1, \sigma, 1, 0, 0)$ and $S^\omega = C = (1, 1, \sigma, 0, 0, 0)$. As before, we can assume that $\mathbb{P}$ has equations $X_4 = X_5 = 0$ and that l has equations $X_0 = X_1 = X_2 = X_3 = 0$. By (ii), if r_i is a line of $\mathcal{L}(S)$ then $r_j^{\phi_{r_i}}$ is contained in $\mathcal{L}(S)$ for any line r_j of $\mathcal{L}(S)$. If $R_j = r_j \cap t$, we say that the point $R_j^{c_{r_i}} = r_j^{\phi_{r_i}} \cap t$ is the *conjugate* of R_j with respect to r_i. Since $r_j \cap t' = R_j^\omega$ and $r_j^{\phi_{r_i}} \cap t' = R_j^{c_{r_i}}$, then $r_j^{\phi_{r_i}}$ belongs to $\mathcal{L}(S)$ if and only if $(R_j^{c_{r_i}})^\omega = (R_j^\omega)^{c_{r_i}}$. Again as in the proof of Theorem 4.12 we can suppose that ω has companion automorphism $x \to x^q$. Let $\omega : \underline{x} = (x_0, x_1, x_2, x_3, x_4, x_5) \in \Sigma' \to A\underline{x}^q \in \Sigma'$, where $A = (a_{ij})$, $i, j \in \{0, 1, 2, 3, 4, 5\}$, is a non-singular matrix over $\mathbb{F}_{q^3}$. By requiring that $T^\omega = M$, $R^\omega = N$ and $S^\omega = C$ we get

$$a_{14} = a_{24} = a_{44} = a_{54} = 0 \quad \text{and} \quad a_{04} = a_{34},$$
$$a_{05} = a_{45} = a_{55} = 0, \qquad a_{15} = a_{35}, \qquad a_{25} = \sigma a_{15}$$

and

$$a_{04} = a_{15} = a_{34}.$$

From $M^\omega = M'$ and $N^\omega \in t^\perp \setminus \{M'\}$, it follows

$$a_{10} = a_{13}, \qquad a_{00} + a_{03} = a_{23} + a_{20} = a_{30} + a_{33},$$
$$a_{40} = a_{43}, \qquad a_{50} = a_{53},$$

and

$$a_{41} = a_{40} + \sigma^q a_{42}, \qquad a_{51} = a_{50} + \sigma^q a_{52}.$$

By requiring that $M'^\omega = M^{\omega^2} = T$ and $(0, 1, \sigma, 0, 0, 0)^\omega \in l \setminus \{T\}$ we get

$$a_{03} = a_{00} + a_{02}, \qquad a_{12} = a_{52} = 0, \qquad a_{22} = a_{02}, \qquad a_{32} = a_{02}, \qquad a_{50} = a_{51},$$

and

$$a_{01} = \sigma^q a_{02}, \qquad a_{11} = 0, \qquad a_{00} = a_{30}, \qquad a_{33} = a_{30} + a_{02} = a_{00} + a_{02},$$
$$a_{23} = a_{00} + a_{02} + \sigma a_{10}, \qquad a_{20} = a_{00} + \sigma a_{10},$$

respectively.

Since the line $r = \langle M, M'\rangle$ is contained in $\mathcal{L}(S)$ by the assumption (ii), the equality $(P^\omega)^{c_r} = (P^{c_r})^\omega$ holds for any point $P \in t \setminus \{M\}$; this leads to $a_{02} = a_{10}$. Moreover, since the line $s =$

$\langle N, N^{\omega}\rangle$, where $N^{\omega} = (a_{00} + a_{02}, a_{10}, a_{00} + a_{02} + \sigma a_{10}, a_{00} + a_{02}, 0, 0)$, is in $\mathcal{L}(S)$, from the assumption (ii) it also follows that $(M^{\omega})^{c_s} = (M^{c_s})^{\omega}$, and this condition gives $a_{00} = \xi a_{02}$, where ξ is an element of $\mathbb{F}_{q^3}$ such that $\xi^2 + \xi = \sigma + \sigma^q$. Since $\mathrm{Tr}_{\mathbb{F}_{q^2}}(\sigma + \sigma^q) = 0$, then there exists $b \in \mathbb{F}_{q^3}$ such that $b^2 + b = \sigma + \sigma^q$. Choose $\xi = b + 1$; by requiring that $(1, 0, 0, 0, 0, 0)^{\omega^3} = (1, 0, 0, 0, 0, 0)$ we get $a_{50}a_{02}^q = a_{02}^{q^2}a_{42}^q \in \mathbb{F}_q$ and $a_{40} = b^q a_{50}$. Hence the matrix A becomes

$$A = \begin{pmatrix} a_{02}(b+1) & \sigma^q a_{02} & a_{02} & ba_{02} & 1 & 0 \\ a_{02} & 0 & 0 & a_{02} & 0 & 1 \\ a_{02}(b+\sigma+1) & \sigma^q a_{02} & a_{02} & a_{02}(b+\sigma) & 0 & \sigma \\ a_{02}(b+1) & \sigma^q a_{02} & a_{02} & ba_{02} & 1 & 1 \\ b^q a_{50} & (b^q+\sigma^q)a_{50} & a_{50} & b^q a_{50} & 0 & 0 \\ a_{50} & a_{50} & 0 & a_{50} & 0 & 0 \end{pmatrix}.$$

A direct computation shows that

$$\Sigma = \mathrm{Fix}(\omega) = \left\{\left(x + \frac{a_{02}}{\rho^q}x^q + \frac{a_{02}}{\rho^q}by^q, y + \frac{a_{02}}{\rho^q}y^q, \frac{a_{02}}{\rho^q}x^q + \sigma y + \frac{a_{02}}{\rho^q}(\sigma + b)y^q,\right.\right.$$
$$\left.\left. x + \frac{a_{02}}{\rho^q}x^q + \frac{a_{02}}{\rho^q}by^q + y, \rho x^{q^2}, \rho y^{q^2}\right) \colon x, y \in \mathbb{F}_{q^3}\right\},$$

where $\rho \in \mathbb{F}_{q^3}$ and $\rho^{q^2+q+1} = a_{50}a_{02}^q$. By projecting Σ from the line l onto $\mathbb{P}$ we get the $\mathbb{F}_q$-linear set

$$L(S) = \left\{\left(x + \frac{a_{02}}{\rho^q}x^q + \frac{a_{02}}{\rho^q}by^q, y + \frac{a_{02}}{\rho^q}y^q, \frac{a_{02}}{\rho^q}x^q + \sigma y + \frac{a_{02}}{\rho^q}(\sigma + b)y^q,\right.\right.$$
$$\left.\left. x + \frac{a_{02}}{\rho^q}x^q + \frac{a_{02}}{\rho^q}by^q + y\right) \colon x, y \in \mathbb{F}_{q^3}\right\},$$

which is the $\mathbb{F}_q$-linear set associated with the generalized twisted field $\mathcal{F}_c$ (q even), with $c = \frac{a_{02}}{\rho^q}$ and $\beta = q$. Finally, if $\xi = b$, using the same arguments as in the previous case, we get the $\mathbb{F}_q$-linear set associated with the generalized twisted field $\mathcal{F}_c$ (q even), with $c = \frac{a_{02}}{\rho^q}$ and $\beta = q^4$. $\square$

Remark 4.14. If S is a generalized twisted field in $\mathcal{F}_5$ and $\mathcal{L}(S)$ is the associated $\mathbb{F}_q$-pseudoregulus, its transpose $\hat{S}$ belongs to $\mathcal{F}_5$ as well. Indeed, the transpose operation corresponds, in the geometric model of linear sets, to a collineation ψ of $\mathbb{P}$ preserving the quadric $\mathcal{Q}$ and interchanging its reguli; this implies that the $\mathbb{F}_q$-pseudoregulus associated with $\hat{S}$ is precisely $\mathcal{L}(S)^{\psi}$. Since the above properties of Theorems 4.12 and 4.13 are preserved under ψ, then the transpose $\hat{S}$ of a generalized twisted field S is isotopic to a generalized twisted field.

5. Final remarks

The partition of the semifields 2-dimensional over their left nucleus and 6-dimensional over their center into non-isotopic families, through the characterization of the associated linear sets (Theorem 4.3), may represent a first step towards their classification.

Such a partition can suggest where one should search to find new examples of semifields with these parameters. Indeed, computational results show that, for fields of small order, there exist examples of such semifields belonging to the family $\mathcal{F}_4$, which are not isotopic to the cyclic ones.

Moreover, the existence of semifields belonging to the family $\mathcal{F}_3$ is still an open problem.

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